Torsion-free metabelian groups with commutator quotient $C_{p^n} \times C_{p^m}$

Carlos Frederico Cid
AEP - Accelerated Encryption Processing
Bray Business Park
Bray, Co. Wicklow - IRELAND
e-mail: carlos.cid@aep.ie

Torsion-free metabelian groups with commutator quotient $C_{p^n} \times C_{p^m}$

Carlos Frederico Cid
AEP - Accelerated Encryption Processing
e-mail: carlos.cid@aep.ie

1 Introduction

Let G be a finitely generated torsion-free metabelian group with finite commutator quotient. Then G is a Bieberbach group, that is, G is a torsion-free group containing a normal, maximal abelian subgroup V of finite rank and index. The subgroup V and the quotient G/V are known as the *translation subgroup* and the *point-group (or holonomy group)* of G, respectively. It is well known that the finiteness of the commutator quotient of G is equivalent to the triviality of the centre of G [6]. In Theorem A. of [3], we showed that every Bieberbach group with finite commutator quotient and point-group isomorphic to $C_{p^n} \times C_{p^m}$ contains a subgroup isomorphic to a torsion-free quotient of

$$K(p^n, p^m) = \left\langle a, b \mid (a^{p^n})^{t(p^m,b)}, (b^{p^m})^{t(p^n,a)}, \left[[a,b], a^{p^n}\right], \left[[a,b], b^{p^m}\right], \ metabelian \right\rangle,$$

where $t(s,x) = \sum_{i=0}^{s-1} x^i$ and the presentation is written relative to the variety of metabelian groups. Furthermore, we showed that $K(p^n, p^m)$ is itself a Bieberbach group of dimension $p^{n+m} - 1$, with point-group $C_{p^n} \times C_{p^m}$ and commutator quotient $C_{p^{n+m}} \times C_{p^{n+m}}$.

In [5], Gupta and Sidki study the existence of torsion-free metabelian groups with a finite elementary abelian p-group as commutator quotient. In particular, they showed that $K(p,p)$ has no proper torsion-free quotients and proved the following theorem.

Theorem 2 of [5] *Let G be a metabelian group such that G/G' is a finite p-group for some prime p. Suppose furthermore that H is a subgroup of G such that $G = G'H$. Then $H' = G' \cap H$.*

They applied the Theorem above and the fact that $K(p,p)$ has no proper torsion-free quotients to show that a finitely generated torsion-free metabelian group can not have commutator quotient isomorphic to $C_p \times C_p$, p prime [5]. On working with the torsion-free quotients of $K(p^n, p^m)$, we are able to investigate the possibilities for a 2-generated abelian p-group to be the commutator quotient of a finitely generated torsion-free metabelian group. In Section 2 we introduce the tools in order to study such quotients. In Section 3, considering the quotients of $K(p, p^m)$, we prove

Theorem A. *There exists a finitely generated torsion-free metabelian group G with commutator quotient isomorphic to $C_{p^n} \times C_{p^m}$ if and only if $n, m \geq 2$.*

In Section 4 we describe the calculations to obtain the torsion-free quotients of $K(p, p^2)$. Furthermore, we present the results obtained in [4] for the groups $K(2, 8)$ and $K(4, 4)$. Using the list of torsion-free quotients of $K(4, 4)$ we obtain

Theorem B. *Let G be a finitely generated, torsion-free metabelian group, with commutator quotient isomorphic to $C_4 \times C_4$. Then G is isomorphic to*

$$K(2,2) = \left\langle a, b \mid (a^2)^{1+b}, (b^2)^{1+a}, \left[[a,b], a^2\right], \left[[a,b], b^2\right], metabelian \right\rangle,$$

the fundamental group of the Hantzsche-Wendt manifold.

2 The group $K(p^n, p^m)$

We recall the notation introduced in [3]. Let

$$F_n = \langle x_1, \ldots, x_n | metabelian \rangle$$

denote the free group of rank n in the variety of metabelian groups. A finitely generated metabelian group G is presented as

$$G = \langle x_1, \ldots, x_n | R_1, R_2, \ldots, R_s, metabelian \rangle \cong F_n / \langle R_1, R_2, \ldots, R_s \rangle^{F_n}.$$

We define the following polynomials, for $s \in \mathbb{N}$:

$$\begin{aligned}
&t(s, x) = 1 + x + \ldots + x^{s-1} \\
&d(x) = x - 1 \\
&l(s, x) = (t(s, x) - s)/d(x) = \textstyle\sum_{i=1}^{s-1} t(i, x) = \sum_{i=0}^{s-2} (s - i - 1) x^i.
\end{aligned}$$

If $g, x_1, \ldots, x_n$ are elements of a group G, and $s_1, \ldots, s_n \in \mathbb{Z}$, then we write

$$g^{s_1 x_1 + s_2 x_2 + \ldots + s_n x_n}$$

for the element $(g^{s_1})^{x_1} (g^{s_2})^{x_2} \ldots (g^{s_n})^{x_n}$.

Whenever it is convenient, we will write additively in abelian subgroups of G. When the commutator subgroup G' of G is abelian, using the module notation, we write

$$[x_1, x_2^s] = [x_1, x_2].t(s, x_2).$$

Consider then

$$K(p^n, p^m) = \left\langle a, b \mid (a^{p^n})^{t(p^m, b)}, (b^{p^m})^{t(p^n, a)}, \left[[a,b], a^{p^n}\right], \left[[a,b], b^{p^m}\right], \ metabelian \right\rangle.$$

We recall that the group $G = K(p^n, p^m)$ is a Bieberbach group of dimension $p^{n+m} - 1$, with point-group isomorphic to $C_{p^n} \times C_{p^m}$ and commutator quotient $C_{p^{n+m}} \times C_{p^{n+m}}$. The commutator subgroup G' of G is free abelian of rank $p^{n+m} - 1$, and if we denote the commutator $[a, b]$ by c and the action of a and b on G' by A and B, respectively, it follows that G' is freely generated by the set

$$\{c.A^i B^j, 0 \leq i < p^n, 0 \leq j < p^m, (i, j) \neq (p^n - 1, p^m - 1)\}.$$

Furthermore $V = \langle a^{p^n}, b^{p^m}, G' \rangle$ is the translation subgroup of G.

Lemma 2.1 *Let M be the $\mathbb{Q}[\frac{G}{V}]$-module defined as $M = \mathbb{Q} \otimes V$. Then M decomposes as a direct sum of*

$$(m - n)p^n + (p + 1)\frac{p^n - 1}{p - 1}$$

irreducible, non-isomorphic submodules.

Proof. It is clear that as $\mathbb{Q}[\frac{G}{V}]$-module, M is cyclic and it is generated by c. And since for $s \geq 1$, we have $gcd(d(x), t(p^s, x)) = 1$, we are able to write

$$M = M_1 \oplus M_2 \oplus M_3 \oplus M_4,$$

where

$$\begin{array}{ll} M_1 = M.d(A)d(B), & M_2 = M.t(p^n, A)d(B) \\ M_3 = M.d(A)t(p^m, B), & M_4 = M.t(p^n, A)t(p^m, B). \end{array}$$

Furthermore we have $M.t(p^n, A)d(A) = M.t(p^m, B)d(B) = 0$. Thus the submodule M_4 is central G and is therefore trivial. When $s \geq 2$, the polynomial $t(p^s, x)$ can be factored as $t(p^{s-i}, x)t(p^i, x^{p^{s-i}})$, for $1 \leq i \leq s - 1$. Thus we can write

$$t(p^s, x) = t(p, x)t(p, x^p)t(p, x^{p^2}) \ldots t(p, x^{p^{s-1}}),$$

where all the terms are irreducible over $\mathbb{Q}$. Let U_j be the companion matrix of the polynomial $t(p, x^{p^{j-1}})$ and Id be the identity matrix. Since M is generated by c, we are able to find a basis for M_2 such that $[A] = Id$ and

$$B = \begin{pmatrix} U_1 & & & \\ & U_2 & & \\ & & \ddots & \\ & & & U_m \end{pmatrix}.$$

Similarly, there exists a basis of M_3 such that $[B] = Id$ and

$$A = \begin{pmatrix} U_1 & & & \\ & U_2 & & \\ & & \ddots & \\ & & & U_n \end{pmatrix}.$$

Therefore M_2 and M_3 decompose as

$$M_2 = \bigoplus_{j=1}^{m} M_{2j} \text{ and } M_3 = \bigoplus_{j=1}^{n} M_{3j},$$

where the submodules M_{2j} and M_{3j} have dimension $p^{j-1}(p-1)$. The actions of a and b on these submodules are given by the matrices above.

On M_1, we have that A and B have $t(p^n, x)$ and $t(p^m, x)$ as minimal polynomials, respectively. If we extend the field of rationals $\mathbb{Q}$ by B, we obtain the algebra

$$\mathbb{Q}[B] \cong \bigoplus_{j=1}^{m} \mathbb{Q}[U_j].$$

And if we extend the algebra $\mathbb{Q}[B]$ by A, we have

$$\mathbb{Q}[B][A] \cong \bigoplus_{j=1}^{m} \mathbb{Q}[U_j][A] \cong \bigoplus_{j=1}^{m} \bigoplus_{i=1}^{n} \mathbb{Q}[U_j^B][U_i^A].$$

Now we can verify in a straightforward manner that these submodules decompose as direct sum of iredutible submodules. Furthermore, it should be clear that they are all non-isomorphic. And it follows from Proposition 2.6 de [7], that describes the structure of the algebra $\mathbb{Q}[\frac{G}{V}]$, that the number of irreducible submodules of M is equal to the number of non-trivial cyclic subgroups of $C_{p^n} \times C_{p^m}$. By induction on $(m+n)$, we can show that $C_{p^n} \times C_{p^m}$ has

$$(m-n)p^n + (p+1)\frac{p^n - 1}{p-1}$$

non-trivial cyclic subgroups, and the result follows. ∎

Notice that we have $(b^{p^m})^{d(b)} = [b^{p^m}, b] = e = [a^{p^n}, a] = (a^{p^n})^{d(a)}$. Now, since $\ker(d(B)) = M_3$ and $\ker(d(A)) = M_2$, we have

$$b^{p^m} \in M_3 \text{ and } a^{p^n} \in M_2.$$

Lemma 2.2 *Let G be a Bieberbach group with translation subgroup V. Furthermore let $N_1, N_2 \trianglelefteq G$, such that G/N_1 and G/N_2 are both torsion-free. If $\mathbb{Q} \otimes (N_1 \cap V) \subseteq \mathbb{Q} \otimes (N_2 \cap V)$, then $N_1 \leq N_2$.*

Proof. We denote $\mathbb{Q} \otimes (N_i \cap V)$ by R_i. Since G/N_1 and G/N_2 are torsion-free, $N_1 \cap V$ and $N_2 \cap V$ are both pure submodules of V and

$$N_1 \cap V = R_1 \cap V \subseteq R_2 \cap V = N_2 \cap V.$$

Let $[G : V] = n$. If $x_1 \in N_1$, then $x_1^n \in N_1 \cap V \subseteq N_2 \cap V$. Since G/N_2 is torsion-free and $x_1^n \in N_2$, we must have $x_1 \in N_2$ and $N_1 \leq N_2$. ∎

We describe now the method we use to compute the torsion-free quotients of $K(p^n, p^m)$. Let N be a non-trivial normal subgroup of $K(p^n, p^m)$. Then the module $R = \mathbb{Q} \otimes (N \cap V)$ is a non-trivial submodule of M. Since M is direct sum of

$$(m-n)p^n + (p+1)\frac{p^n - 1}{p-1} = k$$

irreducible, non-isomorphic submodules, it follows from the Krull-Schmidt Theorem that R is equal to the sum of some of them. Thus we have $2^k - 1$ cases for R to study (we exclude the trivial one).

Suppose that for a certain possibility for R, we find $N \trianglelefteq K(p^n, p^m)$ and $x \in K(p^n, p^m)$, such that $R = \mathbb{Q} \otimes (N \cap V)$ and $x \notin N$, but with $x^s \in N$, $s \geq 2$. Then $K(p^n, p^m)/N$ is not torsion-free but we can define $\overline{N}$ as the normal closure on $K(p^n, p^m)$ of the subgroup $\langle N, x \rangle$ and repeat the analysis with the subgroup $\overline{N}$. It is clear that we might have $\overline{R} = \mathbb{Q} \otimes (\overline{N} \cap V)$ different of R. Also, if x is one of the generators of $K(p^n, p^m)$, then the group $K(p^n, p^m)/\overline{N}$ is cyclic and finite. For instance, we have seen that $a^{p^n} \in M_2$ and $b^{p^m} \in M_3$. Therefore, neither M_2 nor M_3 can be contained in R, in order to obtain a torsion-free quotient. We should look for powers of the generators to eliminate some possibilities for R. Furthermore, it follows from Lemma 2.2 that for any possibility for R being analised, there will be at most one possible $N \trianglelefteq K(p^n, p^m)$, such that $\mathbb{Q} \otimes (N \cap V) = R$ and $K(p^n, p^m)/N$ is torsion-free.

If we denote by $\Lambda_{p,n,m}$ the set of representatives of isomorphism types of torsion-free quotients of $K(p^n, p^m)$, we can turn $\Lambda_{p,n,m}$ into a partially ordered set if we define for any $Q_1, Q_2 \in \Lambda_{p,n,m}$,

$$Q_1 \geq Q_2 \Longleftrightarrow \exists N \trianglelefteq Q_1 \quad s.t. \quad \frac{Q_1}{N} \cong Q_2.$$

Using this method, we compute in Section 4 the list of torsion-free quotients for the groups $K(p, p^2)$, $K(2, 8)$ and $K(4, 4)$, presenting the lattice of $\Lambda_{p,n,m}$ for the last two cases. In Section 3 we use the torsion-free quotients of $K(p, p^m)$ in order to obtain some general properties of torsion-free metabelian groups with finite commutator quotient. The problem of extending this method to the general case is due to the exponential growth of the possibilities of the $K(p^n, p^m)$-module $R = \mathbb{Q} \otimes (N \cap V)$.

3 Quotients of $K(p, p^m)$

As in the previous Section, let V be the translation subgroup of $K(p, p^m)$ and U_j be the companion matrix of the polynomial $t(p, x^{p^{j-1}})$. We have seen in Lemma 2.1 that $M = \mathbb{Q} \otimes V$ decomposes as a direct sum of $mp + 1$ irreducible, non-isomorphic submodules

$$M = \bigoplus_{i=1}^{p-1} \bigoplus_{j=1}^{m} M_{1j_i} \bigoplus_{j=1}^{m} M_{2j} \bigoplus M_3,$$

where M_{1j_i} has dimension $p^{j-1}(p-1)$, with $[A] = U_j^{ip^{j-1}}$ and $[B] = U_j$. M_{2j} has dimension $p^{j-1}(p-1)$, with $[A] = Id$ and $[B] = U_j$, and M_3 has dimension $p-1$, where $[A] = U_1$ and $[B] = Id$.

Lemma 3.1 *Following the terminology above, we have that*

$$(ab^k)^{p^m} \in M_{11_i},$$

for $1 \le i \le p-1$ *and* $k + i = p^m$, *and*

$$(ab^{kp^{j-1}})^{p^m} \in M_{21} \oplus \ldots \oplus M_{2(j-1)} \oplus M_{1j_i},$$

for $1 \le i \le p-1$, $2 \le j \le m$ *and* $k + i = p^{m-j+1}$.

Proof. We will show that $((ab^k)^{p^m})^{(a-b^r)} = e$ if $k + r = p^m$. Since both $(ab^k)^{p^m}$ and b^{p^m} are contained in V, they must commute. Thus $(ab^k)^{p^m}$ commutes with

$$b^{p^m}(ab^k)^{-1} = b^{p^m-k}a^{-1} = b^r a^{-1},$$

and we have

$$((ab^k)^{p^m})^{(1-b^r a^{-1})} = e.$$

We can conjugate the above expression by a, and we obtain

$$((ab^k)^{p^m})^{(a-b^r)} = e$$

if $k + r = p^m$.

Now let $r = ip^{j-1}$, where $1 \le i \le p-1$. By the decomposition we obtained for M, we have

$$\ker(A - B^{ip^{j-1}}) = M_{21} \oplus \ldots \oplus M_{2(j-1)} \oplus M_{1j_i}$$

when $2 \le j \le m$, and

$$\ker(A - B^i) = M_{11_i}$$

when $j = 1$. In fact, A acts as $B^{ip^{j-1}}$ on M_{1j_i} and as Id on M_{2s}, $1 \le s \le m$. Furthermore, B acts as the companion matrix of $t(p, x^{p^{s-1}})$ on M_{2s}. Therefore, for $1 \le s \le j-1$, $B^{ip^{j-1}}$ also acts as Id.

Thus we have

$$(ab^k)^{p^m} \in \ker(A - B^i) = M_{11_i}$$

for $1 \le i \le p-1$ and $k + i = p^m$, and

$$(ab^{kp^{j-1}})^{p^m} \in \ker(A - B^{ip^{j-1}}) = M_{21} \oplus \ldots \oplus M_{2(j-1)} \oplus M_{1j_i}$$

for $1 \le i \le p-1$, $2 \le j \le m$ and $k + i = p^{m-j+1}$. ∎

Remark : Notice that from the factorization of the polynomial $t(p^s, x)$ as

$$t(p^s, x) = t(p^{s-i}, x)t(p^i, x^{p^{s-i}}),$$

we can conclude that the group $K(p^n, p^m)$ has a torsion-free quotient isomorphic to $K(p^{n'}, p^{m'})$, for any $1 \le n' \le n$ and $1 \le m' \le m$.

Proposition 3.2 *For any $2 \leq i, j \leq m+1$, the group $K(p, p^m)$ has a torsion-free quotient with commutator quotient isomorphic to $C_{p^i} \times C_{p^j}$.*

Proof. We use induction on m. If $m = 1$, then $i = j = 2$ and the Proposition is true, since $K(p, p)$ itself has commutator quotient isomorphic to $C_{p^2} \times C_{p^2}$.

Let $m \geq 2$. Since $K(p, p^m)$ has a torsion-free quotient isomorphic to $K(p, p^{m-1})$, by induction we have that $K(p, p^m)$ has a torsion-free quotient with commutator quotient isomorphic to $C_{p^i} \times C_{p^j}$, for all $2 \leq i, j \leq m$. Because the commutator quotient of $K(p, p^m)$ is isomorphic to $C_{p^{m+1}} \times C_{p^{m+1}}$, we have only to find $N_k \trianglelefteq K(p, p^m)$, such that $K(p, p^m)/N_k$ is torsion-free with commutator quotient $C_{p^k} \times C_{p^{m+1}}$, for $2 \leq k \leq m$.

For $2 \leq k \leq m$, let N_k be the normal closure on $K(p, p^m)$ of the subgroup generated by

$$(a^p)^{t(p^{k-1}, b^{p^{m+1-k}})}.$$

It is clear that N_k is contained in the translation subgroup V of $K(p, p^m)$, and $K(p, p^m)/N_k$ has commutator quotient $C_{p^k} \times C_{p^{m+1}}$. Then it remains to show that $K(p, p^m)/N_k$ is torsion-free.

We have seen that $a^p \in M_2$ and that B acts as the companion matrix of $t(p, x^{p^{j-1}})$ on M_{2j}. Since $t(p^{k-1}, b^{p^{m+1-k}})$ can be factored as

$$t(p^{k-1}, b^{p^{m+1-k}}) = t(p, b^{p^{m+1-k}}) \ldots t(p, b^{p^{m-2}}) t(p, b^{p^{m-1}}),$$

we have

$$\ker(t(p^{k-1}, B^{p^{m+1-k}})) = M_{2(m+2-k)} \oplus \ldots \oplus M_{2m},$$

and the element $(a^p)^{t(p^{k-1}, b^{p^{m+1-k}})}$ is contained in $M_{21} \oplus M_{22} \oplus \ldots \oplus M_{2(m+1-k)}$, with non-trivial components in all these submodules. Therefore

$$\mathbb{Q} \otimes N_k = M_{21} \oplus M_{22} \oplus \ldots \oplus M_{2(m+1-k)}$$

and N_k has rank $\sum_{i=0}^{m-k} p^i(p-1) = p^{m+1-k} - 1$.

Then consider

$$\begin{aligned}
(a^p)^{t(p^{k-1}, b^{p^{m+1-k}})} &= a^p (a^p)^{b^{p^{m+1-k}}} \ldots (a^p)^{(b^{p^{m+1-k}})^{p^{k-1}-1}} \\
&= p^{k-1} a^p + c.t(p, A) t(p^{m+1-k}, B) l(p^{k-1}, B^{p^{m+1-k}}) \\
&= p^{k-1} a^p + c.(1 + \ldots + A^{p-1})(1 + \ldots + B^{p^{m+1-k}-1})((p^{k-1} - 1) + \\
&+ \ldots + (B^{p^{m+1-k}})^{p^{k-1}-2}).
\end{aligned}$$

It is clear that the set

$$\{(a^p)^{t(p^{k-1}, b^{p^{m+1-k}}) b^i}, 0 \leq i \leq p^{m+1-k} - 2\}$$

is a basis of N_k. Therefore the elements of N_k can be expresssed as

$$(a^p)^{t(p^{k-1}, b^{p^{m+1-k}}) f(b)},$$

where $f(b) \in \mathbb{Z}[b]$, of degree at most $p^{m+1-k} - 2$. If we compute the Smith Normal Form for the matrix of generators of V/N_k, we can verify in a straightforward manner that V/N_k is torsion-free. We illustrate these calculations with the group $K(2, 8)$ and with N_2 being the normal closure on $K(2, 8)$ of the subgroup generated by $(a^2)^{t(2,b^4)} = (a^2)^{1+b^4}$.

The subgroup N_2 is abelian of rank 3, with free generators

$$(a^2)^{1+b^4}, (a^2)^{b+b^5}, (a^2)^{b^2+b^6}.$$

If we write the elements above in terms of the basis of V, and construct the matrix of generators of V/N_2, we get

$$\begin{pmatrix} 2 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Notice that the last non-zero entry in the last row is equal to 1, and is contained in a column that has all other entries equal to zero. Therefore we can perform elementary column operations and obtain a new matrix, whose last row has only one non-zero entry, which is equal to 1, with all the other rows remaining unchanged. Then we can repeat this procedure with the other rows, until we reach a matrix, equivalent to the above, of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus V/N_2 is torsion-free. The general case is similar to the above, with the rows of the matrix of generators of V/N_k presenting the same characteristics as of the one above, which allows us in the same manner to conclude that V/N_k is torsion-free. Therefore, to show that $K(p, p^m)/N_k$ is torsion-free, it remains to show that there exists no $g \in K(p, p^m)\backslash V$, such that $g^{p^m} \in N_k$.

We should recall that the elements $p^m a^p$ and pb^{p^m} are contained in the commutator subgroup of $K(p, p^m)$, and therefore these can be expressed in terms of the basis of $K(p, p^m)'$. Indeed, we have $p^m a^p = -c.t(p, A)l(p^m, B)$ and $pb^{p^m} = c.t(p^m, B)l(p, A)$, and we can write

$$c.A^{p-1}B^{p^m-2} = -p^m a^p - c.(t(p, A)l(p^m, B) - A^{p-1}B^{p^m-2})$$

and

$$c.A^{p-2}B^{p^m-1} = pb^{p^m} - c.(t(p^m, B)l(p, A) - A^{p-2}B^{p^m-1}).$$

Every $g \in K(p, p^m)$ can be written as $g = a^i b^j v$, where $0 \leq i \leq p-1$, $0 \leq j \leq p^m - 1$ and $v \in V$. Then

$$\begin{aligned} g^{p^m} &= (a^i b^j v)^{p^m} \\ &= ip^{m-1}a^p + jb^{p^m} - c.t(j, B)t(i, A) \sum_{k=1}^{p^m-1} t(k, A^i)B^{jk} + v.t(p^m, A^i B^j) \end{aligned}$$

and we should verify if the equation

$$ip^{m-1}a^p + jb^{p^m} - c.t(j,B)t(i,A)\sum_{k=1}^{p^m-1} t(k,A^i)B^{jk} + v.t(p^m, A^iB^j) =$$
$$= p^{k-1}(a^p)^{f(b)} + c.t(p,A)t(p^{m+1-k},B)l(p^{k-1},B^{p^{m+1-k}})f(B)$$

has non-trivial solutions. Since $f(b)$ has degree at most $p^{m+1-k}-2$, the term in $(a^p)^{t(p^{k-1},b^{p^{m+1-k}})f(b)}$ with the highest sum of exponents would be $c.A^{p-1}B^{p^m-3}$, and therefore the term b^{p^m} will not appear in the expression

$$p^{k-1}(a^p)^{f(b)} + c.t(p,A)t(p^{m+1-k},B)l(p^{k-1},B^{p^{m+1-k}})f(B).$$

Then p must divide j, since if the term b^{p^m} appears in the expression

$$-c.t(j,B)t(i,A)\sum_{k=1}^{p^m-1} t(k,A^i)B^{jk} + v.t(p^m, A^iB^j),$$

its coefficient would be a multiple of p. Therefore $g = a^ib^{j'p}v$ and

$$g^{p^{m-1}} = ip^{m-2}a^p + j'b^{p^m} - c.t(j'p,B)t(i,A)\sum_{k=1}^{p^{m-1}-1} t(k,A^i)B^{j'kp} + v.t(p^{m-1}, A^iB^{j'p}) \in V.$$

We can repeat the argument above m times and conclude that p^m divides j. Then $g = a^iv'$, with $v' \in V$, and $g^p \in V$. Thus the equation can be written for g^p as

$$g^p = ia^p + v'.t(p,A^i) = p^{k-1}(a^p)^{f(b)} + c.t(p,A)t(p^{m+1-k},B)l(p^{k-1},B^{p^{m+1-k}})f(B),$$

and using the same argument, now with a^p, we finally conclude that p divides i and thus arrive at $g \in V$. Thus $K(p,p^m)/N_k$ is torsion-free, of dimension

$$rk(V) - rk(N_k) = p^{m+1} - 1 - (p^{m+1-k}-1) = p^{m+1} - p^{m+1-k} = p^{m+1-k}(p^k-1).$$

The quotient $K(p,p^m)/N_k$ has also point-group isomorphic to $C_p \times C_{p^m}$, since it is not isomorphic to a quotient of $K(p,p^{m-1})$. ∎

Remark : We have seen that the group $K(p^n,p^m)$ has a torsion-free quotient isomorphic to $K(p^{n'},p^{m'})$, for any $1 \le n' \le n$ and $1 \le m' \le m$. In particular, when working with the group $K(p,p^m)$, we have that if $N_{m'}$ is the normal closure on $K(p,p^m)$ of the subgroup

$$\langle (a^p)^{t(p^{m'},b)}, (b^{p^{m'}})^{t(p,a)}, [c, b^{p^{m'}}] \rangle,$$

then $K(p,p^m)/N_{m'} \cong K(p,p^{m'})$. In this case, for $1 \le m' \le m-1$, we have

$$R_{m'} = \mathbb{Q} \otimes (N_{m'} \cap V) = \bigoplus_{i=1}^{p-1} \bigoplus_{j=m'+1}^{m} M_{1j_i} \bigoplus_{j=m'+1}^{m} M_{2j}.$$

Proposition 3.3 *The group $K(p,p^m)$ has no torsion-free quotient with commutator quotient isomorphic to $C_p \times C_{p^m}$.*

Proof. Let V be the translation subgroup of $K(p,p^m)$, M be the module $\mathbb{Q}\otimes V$ and N be a non-trivial normal subgroup of $K(p,p^m)$. Then $R = \mathbb{Q}\otimes(N\cap V)$ is a non-trivial submodule of M, and it should be the sum of some of the $mp+1$ submodules obtained in the decomposition of M. Suppose that $K(p,p^m)/N$ is torsion-free. It follows from Lemma 3.1 that $M_3, M_{11_i} \not\subseteq R$. We divide the possibilities for R in 2 cases.

First suppose that $M_{21} \not\subseteq R$. Then it follows from Lemma 2.2 and the previous remark that $K(p,p^m)/N$ has a torsion-free quotient isomorphic to $K(p,p)$. Since $K(p,p)$ has commutator quotient isomophic to $C_{p^2}\times C_{p^2}$, it is clear that $K(p,p^m)/N$ can not have commutator quotient isomorphic to $C_p \times C_{p^m}$.

Suppose now that $M_{21} \subseteq R$. If $m=1$, then $K(p,p^m)/N$ is not torsion-free, since $a^p \in M_{21}$. Consider then $m \geq 2$. We ask which of the submodules M_{12_i}, M_{22} are contained in R. It follows from Lemma 3.1 that M_{12_i} can not be contained in R, since $(ab^{kp})^{p^m} \in M_{21}\oplus M_{12_i}$ for $1 \leq i \leq p-1$ and $k+i = p^{m-1}$. If $M_{22}\subseteq R$, we repeat this analisys, this time with the submodules M_{13_i}, M_{23}, and so on. Since $a^p \in M_2$, there exists $2 \leq s \leq m$, such that $M_{21}\oplus\ldots\oplus M_{2(s-1)} \subseteq R$, and $M_{1s_i}, M_{2s} \not\subseteq R$, for $1 \leq i \leq p-1$.

Now we apply Proposition 3.2 to the group $K(p,p^s)$. If N_2 is the normal closure on $K(p,p^s)$ of the subgroup generated by $(a^p)^{t(p,b^{p^{s-1}})}$, then $H = K(p,p^s)/N_2$ is torsion-free and has commutator quotient isomorphic to $C_{p^2}\times C_{p^{s+1}}$. However it follows again from Lemma 2.2 and the previous remark that $K(p,p^m)/N$ has a torsion-free quotient isomorphic to H, and therefore can not have commutator quotient $C_p \times C_{p^m}$. ■

We are now able to prove Theorem A.

Theorem A. *There exists a finitely generated torsion-free metabelian group G with commutator quotient isomorphic to $C_{p^n}\times C_{p^m}$ if and only if $n,m \geq 2$.*

Proof. By the result of Proposition 3.2, it remains to show that there is no finitely generated, torsion-free metabelian group with commutator quotient isomorphic to $C_p \times C_{p^m}$, for $m \geq 1$. Suppose that there exists a metabelian group of this type. If $x,y \in G$ are the generators of G modulo G' and $H = \langle x,y\rangle$, then it follows from Theorem 2 of [5] that H is a 2-generated, metabelian Bieberbach group, with

$$\frac{H}{H'} \cong \frac{G}{G'} \cong C_p \times C_{p^m}.$$

Furthermore, if we denote by V_H the translation subgroup of H, we have that $H' \leq V_H$, and Theorem A. of [3] tells us that H is isomorphic to a torsion-free quotient of $K(p,p^m)$. However, it follows from the previous Proposition that $K(p,p^m)$ does not have a torsion-free quotient of this type, and we reach a contradiction. ■

We now compute the torsion-free quotients for some other groups $K(p^n, p^m)$, using the method described in Section 2. We illustrate this method with the calculations for $K(p, p^2)$. In [4], one can find the calculations for $K(2, 8)$ and $K(4, 4)$.

4 Torsion-free quotients of $K(p, p^2)$

Let

$$K(p, p^2) = \left\langle a, b \mid (a^p)^{t(p^2,b)}, (b^{p^2})^{t(p,a)}, [[a, b], a^p], \left[[a, b], b^{p^2}\right], \ metabelian \right\rangle.$$

The group $K(p, p^2)$ is a Bieberbach group of dimension $p^3 - 1$, with point-group isomorphic to $C_p \times C_{p^2}$ and commutator quotient isomorphic to $C_{p^3} \times C_{p^3}$. Let V denote once more the translation subgroup of $G = K(p, p^2)$ and $c = [a, b]$. It follows from Section 2 that the module $M = \mathbb{Q} \otimes V$ decomposes as a sum of $2p+1$ irreducible, non-isomorphic submodules

$$M = \bigoplus_{i=1}^{p-1} \bigoplus_{j=1}^{2} M_{1j_i} \bigoplus_{j=1}^{2} M_{2j} \bigoplus M_3,$$

where M_{11_i}, M_{21}, M_3 have dimension $p - 1$ and M_{12_i}, M_{22} have dimension $p(p - 1)$. The actions of a and b on these submodules were described in the previous Section.

We have that $b^{p^2} \in M_3$ and $a^p \in M_2$. Furthermore, $(a^p)^{t(p,b^p)} \in M_{21}$ and $(a^p)^{t(p,b)} \in M_{22}$, and both are non trivial. It follows from Lemma 3.1 that

$$(ab^k)^{p^2} \in M_{11_i}$$

for $1 \leq i \leq p - 1$ and $k + i = p^2$, and

$$(ab^{kp})^{p^2} \in M_{21} \oplus M_{12_i}$$

for $1 \leq i \leq p - 1$ and $k + i = p$. For the last case, we have

$$\begin{aligned}
((ab^{kp})^p)^{t(p,b^p)} &= (a^p)^{t(p,b^p)} + kpb^{p^2} - c.t(kp, B) \sum_{i=1}^{p-1} t(i, A)(B^{kp})^i t(p, B^{kp}) \\
&= (a^p)^{t(p,b^p)} + kpb^{p^2} \\
&\quad -c.(1 + B^p + \ldots + B^{p(k-1)})t(p, B)t(p, B^p) \sum_{i=1}^{p-1} t(i, A)(B^{kp})^i \\
&= (a^p)^{t(p,b^p)} + kpb^{p^2} - kc.t(p^2, B)l(p, A) \\
&= (a^p)^{t(p,b^p)} \in M_{21},
\end{aligned}$$

and $0 \neq ((ab^{kp})^p)^{t(p,b)} \in M_{12_i}$.

Lemma 4.1 *For $1 \le i \le p-1$, we have $(b^p)^{t(p,a^i)} \in M_{22} \oplus M_{12_k}$, where $ik \equiv 1 mod\, p$.*

Proof. On writing additively, we have

$$(b^p)^{t(p,a^i)} = b^{p^2} - c.t(p,B)t(i,A)\sum_{j=1}^{p-1} t(j,A^i)B^{p(p-1-j)}.$$

If we show that $(b^p)^{t(p,a^i)(a-1)(a-b^{kp})} = 0$, then $(b^p)^{t(p,a^i)} \in M_{21} \oplus M_{22} \oplus M_{12_k}$ would follow. First we calculate

$$\begin{aligned} (b^p)^{t(p,a^i)(a-1)} &= (b^{p^2})^{a-1} - c.t(p,B)(A^i-1)\textstyle\sum_{j=1}^{p-1} t(j,A^i)B^{p(p-1-j)} \\ &= -c.t(p^2,B) - c.t(p,B)\textstyle\sum_{j=1}^{p-1}(A^{ij}-1)B^{p(p-1-j)} \\ &= -c.t(p^2,B) + c.t(p,B)t(p,B^p) - c.t(p,B)\textstyle\sum_{j=0}^{p-1} A^{ij}B^{p(p-1-j)} \\ &= -c.t(p,B)\textstyle\sum_{j=0}^{p-1} A^{ij}B^{p(p-1-j)}. \end{aligned}$$

Now we write $s = p-1-j$. Then

$$\begin{aligned} (b^p)^{t(p,a^i)(a-1)(a-b^{kp})} &= -c.t(p,B)(A-B^{kp})\textstyle\sum_{j=0}^{p-1} A^{ij}B^{p(p-1-j)} \\ &= -c.t(p,B)\textstyle\sum_{s=0}^{p-1} A^{i(p-1-s)+1}B^{ps} + \\ &+c.t(p,B)\textstyle\sum_{s=0}^{p-1} A^{i(p-1-s)}B^{p(s+k)}, \end{aligned}$$

and after reordering the terms of $c.t(p,B)\sum_{s=0}^{p-1} A^{i(p-1-s)}B^{p(s+k)}$, we have

$$\begin{aligned} c.t(p,B)\textstyle\sum_{s=0}^{p-1} A^{i(p-1-s)}B^{p(s+k)} &= c.t(p,B)\textstyle\sum_{s=0}^{p-1} A^{i(p-1-s)}B^{p(s+k)} \\ &= c.t(p,B)\textstyle\sum_{s=0}^{p-1} A^{i(p-1-s+k)}B^{ps} \\ &= c.t(p,B)\textstyle\sum_{s=0}^{p-1} A^{i(p-1-s)+1}B^{ps}, \end{aligned}$$

since $ik \equiv 1 \mathrm{mod}\, p$. Thus $(b^p)^{t(p,a^i)(a-1)(a-b^{kp})} = 0$ and

$$(b^p)^{t(p,a^i)} \in M_{21} \oplus M_{22} \oplus M_{12_k}.$$

To prove that $(b^p)^{t(p,a^i)} \in M_{22} \oplus M_{12_k}$, it is enough to show that $(b^p)^{t(p,a^i)t(p,b^p)} = 0$. Then

$$\begin{aligned} (b^p)^{t(p,a^i)t(p,b^p)} &= pb^{p^2} - c.t(p,B)t(p,B^p)t(i,A)\textstyle\sum_{j=1}^{p-1} t(j,A^i)B^{p(p-1-j)} \\ &= pb^{p^2} - c.t(p^2,B)t(i,A)\textstyle\sum_{j=1}^{p-1} t(j,A^i) \\ &= pb^{p^2} - c.t(p^2,B)t(i,A)l(p,A^i). \end{aligned}$$

Now we have

$$c.l(p,A^i)(A^i-1) = c.(t(p,A^i)-p) = c.(t(p,A)-p) = c.d(A)l(p,A)$$

and therefore

$$c.(l(p,A^i)t(i,A) - l(p,A))d(A) = 0,$$

and $c.l(p, A^i)t(i, A) - c.l(p, A) \in M_2$. Thus

$$c.l(p, A^i)t(i, A) = c.l(p, A) + m_2,$$

where $m_2 \in M_2$. However, since $m_2.t(p^2, B) = 0$, we have

$$\begin{aligned}(b^p)^{t(p,a^i)t(p,b^p)} &= pb^{p^2} - c.t(p^2, B)t(i, A)l(p, A^i) \\ &= pb^{p^2} - c.t(p^2, B)l(p, A) = 0,\end{aligned}$$

and therefore, for $1 \leq i \leq p-1$, we have $(b^p)^{t(p,a^i)} \in M_{22} \oplus M_{12_k}$, where $ik \equiv 1 \text{mod } p$. Furthermore, we can easily verify that the components of it in both submodules are non-trivial. ∎

Proposition 4.2 *The group $K(p, p^2)$ has $\frac{2^p-2}{p}+2$ proper, non-isomorphic torsion-free quotients.*

Proof. Let $N \trianglelefteq G = K(p, p^2)$. Then $R = \mathbb{Q} \otimes (N \cap V)$ is sum of some of the $2p+1$ submodules obtained in the decomposition of M. Therefore we have $2^{2p+1}-1$ cases to study (we exclude the trivial case). It follows from Lemma 2.2 that for any possibility for R being studied, there will be at most one possible $N \trianglelefteq K(p, p^2)$, such that $\mathbb{Q} \otimes (N \cap V) = R$ and $K(p, p^2)/N$ is torsion-free.

Since $b^{p^2} \in M_3$ and $(ab^k)^{p^2} \in M_{11_i}$, where $k + i = p^2$, we have that $M_3, M_{11_i} \not\subseteq R$. Thus we have $2^{p+1}-1$ cases to study. If $M_{21} \subseteq R$, it follows from Lemma 3.1 that no other submodule of M can be contained in R.

Let $N = \langle (a^p)^{t(p,b^p)} \rangle^G$. It is clear that $\mathbb{Q} \otimes N = M_{21}$. Now, in Proposition 3.2 in showed that

$$\frac{G}{N} \cong \left\langle a, b \mid (a^p)^{t(p,b^p)}, (b^{p^2})^{t(p,a)}, [[a, b], a^p], \left[[a, b], b^{p^2}\right], \ metabelian \right\rangle$$

is a Bieberbach group of dimension $p^3 - 1 - p + 1 = p^3 - p$, point-group isomorphic to $C_p \times C_{p^2}$ and commutator quotient $C_{p^2} \times C_{p^3}$. Notice that this group has no proper torsion-free quotients. We denote it by T_M.

Now R can be equal to the sum of any of the submodules M_{12_j} and M_{22}. Thus we have $2^p - 1$ cases to study. Once we find $N \trianglelefteq V$, such that $\mathbb{Q} \otimes N = R$ and $\frac{V}{N}$ is torsion-free, that must be enough, since it follows from Lemma 2.2 and the remark before Proposition 3.3 that the group $\frac{G}{N}$ will have a quotient isomorphic to $K(p, p)$, with the kernel of the epimorphism contained in $\frac{V}{N}$.

Let R be equal to one of these submodules, for instance M_{22}. If $N = \langle (a^p)^{t(p,b)} \rangle^G$, then $\mathbb{Q} \otimes N = M_{22}$ and if we compute the Smith Normal Form for the matrix of generators of $\frac{V}{N}$, we can show in a similar manner to the proof of Proposition 3.2, that $\frac{V}{N}$ is torsion-free. Thus

$$\frac{G}{N} \cong \left\langle a, b \mid (a^p)^{t(p,b)}, (b^{p^2})^{t(p,a)}, [[a, b], a^p], \left[[a, b], b^{p^2}\right], \ metabelian \right\rangle$$

is a Bieberbach group of dimension $p^3-1-p^2+p=(p^2+1)(p-1)$. It also has point-group isomorphic to $C_p\times C_{p^2}$ and commutator quotient isomorphic to $C_{p^2}\times C_{p^3}$. All the remaining cases for R being equal to one of the submodules M_{12_j}, M_{22} is isomophic to the group above, by the isomorphism induced by the automorphism of $K(p,p^2)$ given by $a\mapsto ab^p$, $b\mapsto b$; see [3]. We denote this group by T_1.

Now suppose R is sum of two of the submodules M_{12_j}, M_{22}. If $p=2$, then $\frac{G}{N}\cong K(2,2)$. If p is odd, then we have $\binom{p}{2}$ possibilities in this case, but using once more the isomorphism defined above, we can suppose that $M_{22}\subseteq R$ and we have $\frac{1}{p}\binom{p}{2}=\frac{p-1}{2}$ cases to study. We have seen that

$$(b^p)^{t(p,a^i)}\in M_{22}\oplus M_{12_k},$$

where $ik\equiv 1 \text{mod } p$. Let $N_k=\langle (a^p)^{t(p,b)},(b^p)^{t(p,a^i)}\rangle^G$, for $1\leq k\leq \frac{p-1}{2}$. We can show again that $\frac{V}{N_k}$ is torsion-free, since N_k is a pure submodule of V. And because $\frac{G}{N_k}$ has a quotient isomorphic to $K(p,p)$, with the kernel contained in $\frac{V}{N_k}$, we have that

$$\frac{G}{N_k}\cong\left\langle a,b\mid (a^p)^{t(p,b)},(b^p)^{t(p,a^i)},(b^{p^2})^{t(p,a)},[[a,b],a^p],\left[[a,b],b^{p^2}\right],\ metabelian\right\rangle$$

is a Bieberbach group of dimension $p^3-1-2(p^2-p)$, point-group isomorphic to $C_p\times C_{p^2}$ and commutator quotient $C_{p^2}\times C_{p^2}$. There are $\frac{p-1}{2}$ groups and applying the Theorem 2.2, Chapter III of [2], we can show that they are all non-isomorphic, since there is not a semi-linear homomorphism (f,σ) between their translation subgroups, such that $f(m.A^iB^j)=f(m).\sigma(A^iB^j)$. We denote these groups by T_{21}, T_{22}, $\ldots, T_{2i_2}$, where $i_2=\frac{p-1}{2}$.

If R is equal to sum of n submodules, $3\leq n\leq p-1$, then using the automorphism defined above, we can suppose that $M_{22}\subseteq R$ and there are $\frac{1}{p}\binom{p}{n}=i_n$ non-isomorphic torsion-free quotients (using again Theorem 2.2, Chapter III of [2]), defined as following :

For each n, we obtain R_k, $1\leq k\leq i_n$, and define $N_k=V\cap R_k$. Then N_k is a pure submodule of V and $\frac{V}{N_k}$ is torsion-free, Since $\frac{G}{N_k}$ has quotient isomophic to $K(p,p)$, with kernel contained in $\frac{V}{N_k}$, we have that $\frac{G}{N_k}$ is a Bieberbach group, of dimension $p^3-1-n(p^2-p)$. Furthermore, G/K_k has point-group isomorphic to $C_p\times C_{p^2}$ (otherwise it would be isomorphic to $K(p,p)$) and commutator quotient isomorphic to $C_{p^2}\times C_{p^2}$. Indeed, they are all quotients of some of the T_{2j} defined above and have $K(p,p)$ as quotient. And of course these groups have commutator quotient isomophic to $C_{p^2}\times C_{p^2}$. For each n, we have $i_n=\frac{1}{p}\binom{p}{n}$ quotients, that we denote by $T_{n1},\ldots,T_{ni_n}$.

And finally, if R is sum of p submodules M_{22}, M_{12_j}, we have $\frac{G}{N}$ isomorphic to $K(p,p)$. Thus we have a total of $\frac{2^p-2}{p}+2$ proper, non-isomorphic quotients of $K(p,p^2)$. Notice that T_M and $K(p,p)$ are the only ones that have no proper torsion-free quotient. ■

In particular, when $p = 2$, the group $K(2,4)$ has 3 proper, non-isomorphic torsion-free quotients, given by:

$$
\begin{aligned}
H_1 &= \langle a, b \mid (a^2)^{1+b^2}, (b^4)^{1+a}, [[a,b],b^4], [[a,b],a^2],\ \mathit{metabelian}\rangle \\
H_2 &= \langle a, b \mid (a^2)^{1+b}, (b^4)^{1+a}, [[a,b],b^4], [[a,b],a^2],\ \mathit{metabelian}\rangle \\
K(2,2) &= \langle a, b \mid (a^2)^{1+b}, (b^2)^{1+a}, [[a,b],b^2], [[a,b],a^2],\ \mathit{metabelian}\rangle,
\end{aligned}
$$

where H_1 and H_2 have dimension 6 and 5, respectively. Both have point-group isomorphic to $C_2 \times C_4$ and commutator quotient $C_4 \times C_8$.

We should notice that eventhough for p odd, we found torsion-free quotients with point-group $C_p \times C_{p^2}$ and commutator quotient $C_{p^2} \times C_{p^2}$, this did not happen when $p = 2$.

5 Torsion-free quotients of $K(2,8)$ and $K(4,4)$

Using the method of the previous Section, we are able to produce the following complete list of torsion-free quotients of $K(2,8)$ and $K(4,4)$; the proof can be found in [4].

Proposition 4.4 of [4] *The group $K(2,8)$ has 12 proper, non-isomorphic torsion-free quotients.*

$$
\begin{aligned}
Q_1 &= \langle a, b \mid (a^2)^{(1+b^2)(1+b^4)}, (b^8)^{t(2,a)}, [[a,b],b^8], [[a,b],a^2],\ \mathit{metabelian}\rangle \\
Q_2 &= \langle a, b \mid (a^2)^{(1+b)(1+b^4)}, (b^8)^{t(2,a)}, [[a,b],b^8], [[a,b],a^2],\ \mathit{metabelian}\rangle \\
Q_3 &= \langle a, b \mid (a^2)^{t(8,b)}, (b^8)^{t(2,a)}, ((ab^2)^4)^{1+b}, [[a,b],b^8], [[a,b],a^2],\ \mathit{metabelian}\rangle \\
Q_4 &= \langle a, b \mid (a^2)^{t(4,b)}, (b^8)^{t(2,a)}, [[a,b],b^8], [[a,b],a^2],\ \mathit{metabelian}\rangle \\
Q_5 &= \langle a, b \mid (a^2)^{1+b}, (b^8)^{t(2,a)}, [[a,b],b^8], [[a,b],a^2],\ \mathit{metabelian}\rangle \\
Q_6 &= \langle a, b \mid (a^2)^{1+b^2}, (b^8)^{t(2,a)}, [[a,b],b^8], [[a,b],a^2],\ \mathit{metabelian}\rangle \\
Q_7 &= \langle a, b \mid (a^2)^{1+b^4}, (b^8)^{t(2,a)}, [[a,b],b^8], [[a,b],a^2],\ \mathit{metabelian}\rangle \\
Q_8 &= \langle a, b \mid (a^2)^{t(4,b)}, (b^8)^{t(2,a)}, ((ab^2)^4)^{1+b}, b^8[a,b]^{(1+b)(a-b^4)}, [[a,b],b^8], [[a,b],a^2],\ \mathit{metab.}\rangle \\
Q_9 &= K(2,4) \\
Q_{10} &= H_1 = \langle a, b \mid (a^2)^{1+b^2}, (b^4)^{t(2,a)}, [[a,b],b^4], [[a,b],a^2],\ \mathit{metabelian}\rangle \\
Q_{11} &= H_2 = \langle a, b \mid (a^2)^{1+b}, (b^4)^{t(2,a)}, [[a,b],b^4], [[a,b],a^2],\ \mathit{metabelian}\rangle \\
Q_{12} &= K(2,2).
\end{aligned}
$$

The groups Q_1, Q_2, Q_3 and Q_4 have point-group isomorphic to $C_2 \times C_8$ and commutator quotient $C_8 \times C_{16}$, with dimensions 14, 13, 13 and 11, respectively.

The groups Q_5, Q_6 and Q_7 have point-group isomorphic to $C_2 \times C_8$ and commutator quotient $C_4 \times C_{16}$, with dimensions 9, 10 and 12, respectively.

The group Q_8 has point-group isomorphic to $C_2 \times C_8$, commutator quotient $C_8 \times C_8$ and dimension 9.

The groups Q_9, Q_{10}, Q_{11} and Q_{12} are quotients of $K(2,4)$ and have already been described.

It follows from the lattice of $\Lambda_{2,1,3}$ (Figure 1) that Q_7, H_1 and $K(2,2)$ have no proper torsion-free quotients.

Proposition 4.5 of [4] *The group $K(4,4)$ has 19 proper, non-isomorphic torsion-free quotients.*

$$
\begin{aligned}
S_1 &= \langle a,b \mid (a^4)^{1+b^2}, (b^4)^{t(4,a)}, [[a,b],b^4], [[a,b],a^4],\ \textit{metabelian}\rangle\\
S_2 &= \langle a,b \mid (a^4)^{1+b}, (b^4)^{t(4,a)}, [[a,b],b^4], [[a,b],a^4],\ \textit{metabelian}\rangle\\
S_3 &= \langle a,b \mid (a^4)^{1+b^2}, (b^4)^{1+a^2}, [[a,b],b^4], [[a,b],a^4],\ \textit{metabelian}\rangle\\
S_4 &= \langle a,b \mid (a^4)^{1+b^2}, (b^4)^{1+a}, [[a,b],b^4], [[a,b],a^4],\ \textit{metabelian}\rangle\\
S_5 &= \langle a,b \mid (a^4)^{1+b}, (b^4)^{1+a}, [[a,b],b^4], [[a,b],a^4],\ \textit{metabelian}\rangle\\
S_6 &= \langle a,b \mid (b^4)^{1+a}, ((a^2b)^4)^{1+a}, (a^2)^{(1+b).(1+b^2)}, [[a,b],b^4], [[a,b],a^4],\ \textit{metabelian}\rangle\\
S_7 &= \langle a,b \mid (a^4)^{1+b^2}, (b^4)^{1+a^2}, ((ab)^4)^{1+a^2}, [[a,b],b^4], [[a,b],a^4],\ \textit{metabelian}\rangle\\
S_8 &= \langle a,b \mid (a^4)^{1+b^2}, (b^4)^{1+a^2}, ((ab)^4)^{1+a}, [[a,b],b^4], [[a,b],a^4],\ \textit{metabelian}\rangle\\
S_9 &= \langle a,b \mid (a^4)^{1+b^2}, (b^4)^{1+a}, ((a^2b)^4)^{1+a}, (a^2)^{(1+b).(1+b^2)}, [[a,b],b^4], [[a,b],a^4],\ \textit{metabelian}\rangle\\
S_{10} &= \langle a,b \mid (a^4)^{1+b^2}, (b^4)^{1+a}, ((ab)^4)^{1+a}, [[a,b],b^4], [[a,b],a^4],\ \textit{metabelian}\rangle\\
S_{11} &= \langle a,b \mid (a^4)^{1+b}, (b^4)^{1+a}, ((a^2b)^4)^{1+a}, (a^2)^{(1+b).(1+b^2)}, [[a,b],b^4], [[a,b],a^4],\ \textit{metabelian}\rangle\\
S_{12} &= \langle a,b \mid (a^4)^{1+b}, (b^4)^{1+a}, ((ab)^4)^{1+a}, [[a,b],b^4], [[a,b],a^4],\ \textit{metabelian}\rangle\\
S_{13} &= \langle a,b \mid (a^4)^{1+b}, (b^4)^{1+a}, ((ab^3)^4)^{1+a}, [[a,b],b^4], [[a,b],a^4],\ \textit{metabelian}\rangle\\
S_{14} &= \langle a,b \mid (a^4)^{1+b}, ((ab)^4)^{1+b}, (b^4)^{1+a}, ((a^2b)^4)^{1+a}, (a^2)^{(1+b).(1+b^2)}, [a,b]^{1+b+a^3+ab},\\
&\qquad [[a,b],b^4], [[a,b],a^4],\ \textit{metabelian}\rangle\\
S_{15} &= \langle a,b \mid (a^2)^{1+b^2}, (a^4)^{t(4,b)}, (b^4)^{1+a}, [[a,b],b^4], [[a,b],a^4],\ \textit{metabelian}\rangle\\
S_{16} &= K(2,4) = \langle a,b \mid (a^2)^{t(4,b)}, (b^4)^{1+a}, [[a,b],b^4], [[a,b],a^2],\ \textit{metabelian}\rangle\\
S_{17} &= H_1 = \langle a,b \mid (a^2)^{1+b^2}, (b^4)^{1+a}, [[a,b],b^4], [[a,b],a^2],\ \textit{metabelian}\rangle\\
S_{18} &= H_2 = \langle a,b \mid (a^2)^{1+b}, (b^4)^{1+a}, [[a,b],b^4], [[a,b],a^2],\ \textit{metabelian}\rangle\\
S_{19} &= K(2,2) = \langle a,b \mid (a^2)^{1+b}, (b^2)^{1+a}, [[a,b],b^2], [[a,b],a^2],\ \textit{metabelian}\rangle.
\end{aligned}
$$

The groups S_1 and S_2 have point-group $C_4 \times C_4$, commutator quotient $C_8 \times C_{16}$, and dimensions 14 and 13, respectively.

The groups S_3, S_4, S_5, S_6, S_7, S_8, S_9, S_{10}, S_{11}, S_{12}, S_{13} and S_{14} have all point-group $C_4 \times C_4$ and commutator quotient $C_8 \times C_8$, with dimensions 13, 12, 11, 11, 12, 11, 10, 10, 9, 9, 9 and 7, respectively.

The group S_{15} has point-group isomorphic to $C_4 \times C_4$, commutator quotient $C_4 \times C_8$ and dimension 8.

The groups S_{16}, S_{17}, S_{18} and S_{19} are quotients of $K(2,4)$ and have already been described.

It follows from the lattice of $\Lambda_{2,2,2}$ (Figure 2)that the groups S_7, S_8, H_1 and $K(2,2)$ have no proper torsion-free quotients.

From the list of quotients of $K(4,4)$, we can obtain the following characterization of $K(2,2)$.

Theorem B. *Let G be a finitely generated, torsion-free metabelian group, with commutator quotient isomorphic to $C_4 \times C_4$. Then G is isomorphic to*

$$K(2,2) = \left\langle a, b \mid (a^2)^{1+b}, (b^2)^{1+a}, \left[[a,b], a^2\right], \left[[a,b], b^2\right], metabelian \right\rangle,$$

the fundamental group of the Hantzsche-Wendt manifold.

Proof. Let $a, b \in G$ be the generators of Gmodulo G' and $H = \langle a, b\rangle$. Then $G = HG'$ and it follows from Theorem 2 of [5] that H is a 2-generated torsion-free metabelian group, with

$$\frac{H}{H'} \cong \frac{G}{G'} \cong C_4 \times C_4.$$

Furthermore, both G and H are Bieberbach groups. We denote by V_H the translation subgroup of H. Since $H' \leq V_H$, then it follows from Theorem A of [3] that H is isomorphic to a torsion-free quotient of $K(4,4)$. Now, by the list of torsion-free quotients of $K(4,4)$ given above, the only torsion-free quotient of $K(4,4)$ with commutator quotient isomorphic to $C_4 \times C_4$ is $K(2,2)$. Thus $H \cong K(2,2)$.

Furthermore, we can repeat part of the proof of Proposition 2.3 of [3] and show that $G' = [G', H]H'$. Then we define the normal subgroup $N = (G')^2 H'$, and since G is finitely generated, we have that $\frac{G}{N}$ is a finite 2-group. Now we can compute the second and third terms of the lower central series of $\frac{G}{N}$

$$\Gamma_2\left(\frac{G}{N}\right) = \left[\frac{G}{N}, \frac{G}{N}\right] = \frac{G'N}{N} = \frac{G'}{N}$$

and

$$\Gamma_3\left(\frac{G}{N}\right) = \left[\frac{G'}{N}, \frac{G}{N}\right] = \frac{[G', G]N}{N} = \frac{[G', G'H]N}{N} = \frac{[G', H]H'(G')^2}{N} = \frac{G'}{N}.$$

Thus $\Gamma_2(\frac{G}{N}) = \Gamma_3(\frac{G}{N})$, and because $\frac{G}{N}$ is nilpotent, $G' = N = (G')^2 H'$. Now we can show that

$$dim(H) = rk(H') = rk(G') = dim(G).$$

Thus G is also a 3-dimensional Bieberbach group, with commutator quotient isomorphic to $C_4 \times C_4$. By [1], we have that G is isomorphic to the fundamental group of the Hantzsche-Wendt manifold, that is, $G \cong K(2,2)$. ∎

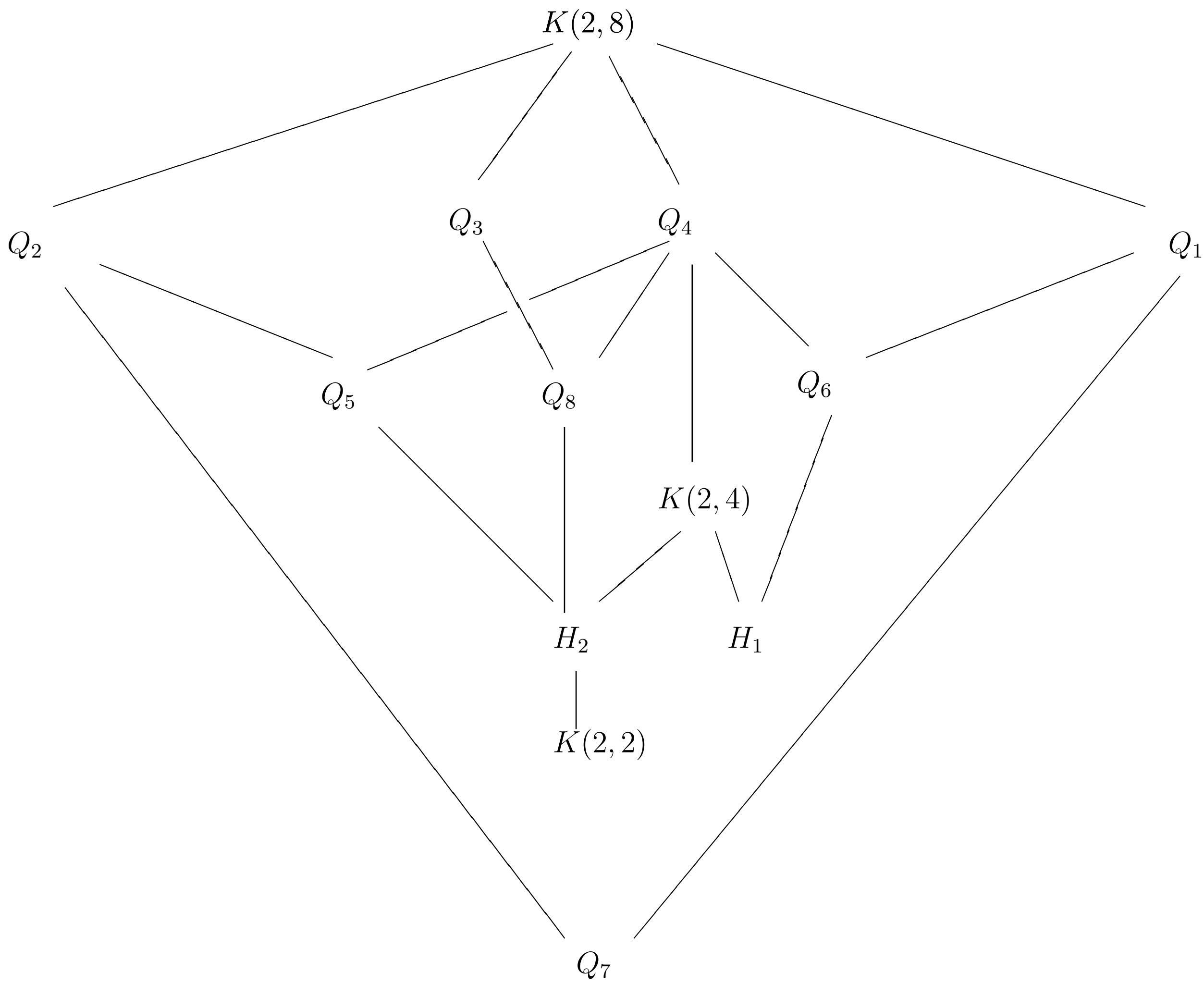

Figure 1: $\Lambda_{2,1,3}$

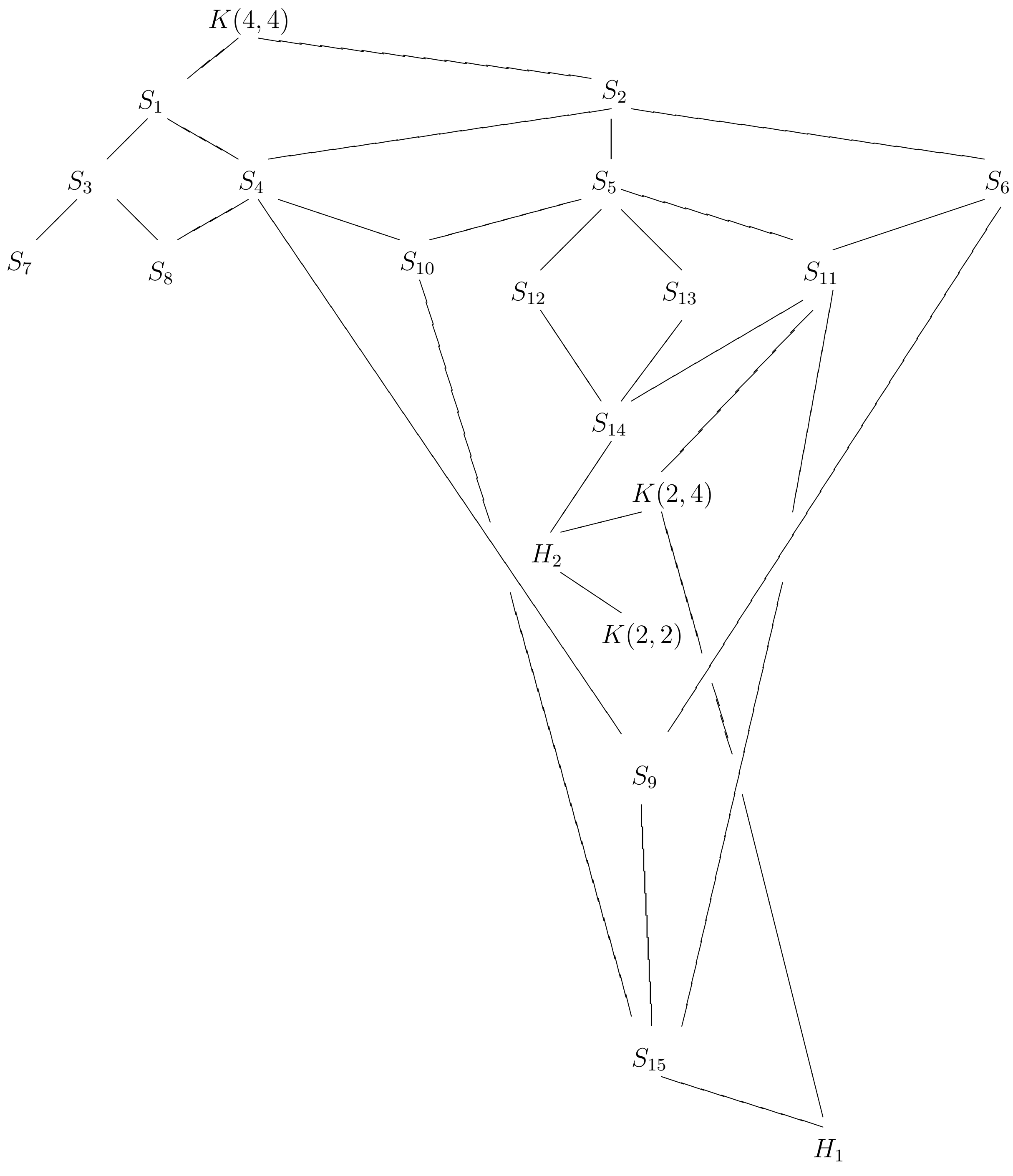

Figure 2: $\Lambda_{2,2,2}$

References

[1] H. Brown, R. Bülow, J. Neubüser, H. Wondratschek and H. Zassenhaus, "Crystallographic groups of four-dimensional space", Wiley, New York, 1978

[2] L. Charlap, "Bieberbach Groups and Flat Manifolds", Springer-Verlag 1986

[3] C. Cid, Bieberbach groups with finite commutator quotient and point-group $C_{p^n} \times C_{p^m}$, *Journal of Group Theory* **3** (2000), 113-125

[4] C. Cid, Tese de Doutorado, Universidade de Brasília, 1999

[5] N. Gupta and S. Sidki, On Torsion-free metabelian Groups with Commutator Quotients of Prime Exponent, *International Journal of Algebra and Computation* Vol. 9, No. 5 (1999), 493-520

[6] H. Hiller and C. Sah, Holonomy of flat manifolds with $b_1 = 0$. *Quart.J.Math.Oxford* **37** (1986), 177-187

[7] S. Sehgal, "Topics in Group Rings", Marcel Dekker, Inc. 1978